Searching for optimal Boolean chains

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Boolean chains

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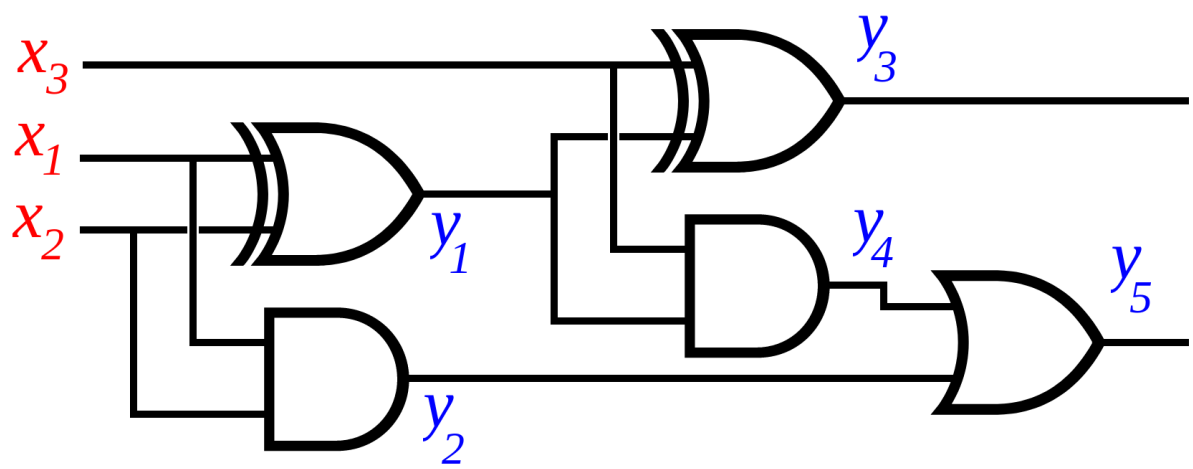
For example, the full adder has $n = 5$ gates and $k = 3$ inputs:

1. $y_1 = x_1 \oplus x_2$;
2. $y_2 = x_1 \wedge x_2$;
3. $y_3 = y_1 \oplus x_3$;
4. $y_4 = y_1 \wedge x_3$;
5. $y_5 = y_2 \vee y_4$.



Each gate can only depend on inputs or previously-computed values.

Five normal gates

Without loss of generality we can assume that all gates $\circ$ are:

- **Nontrivial**: $a \circ b$ depends on both a and b;
- **Zero-preserving**: $0 \circ 0 = 0$.

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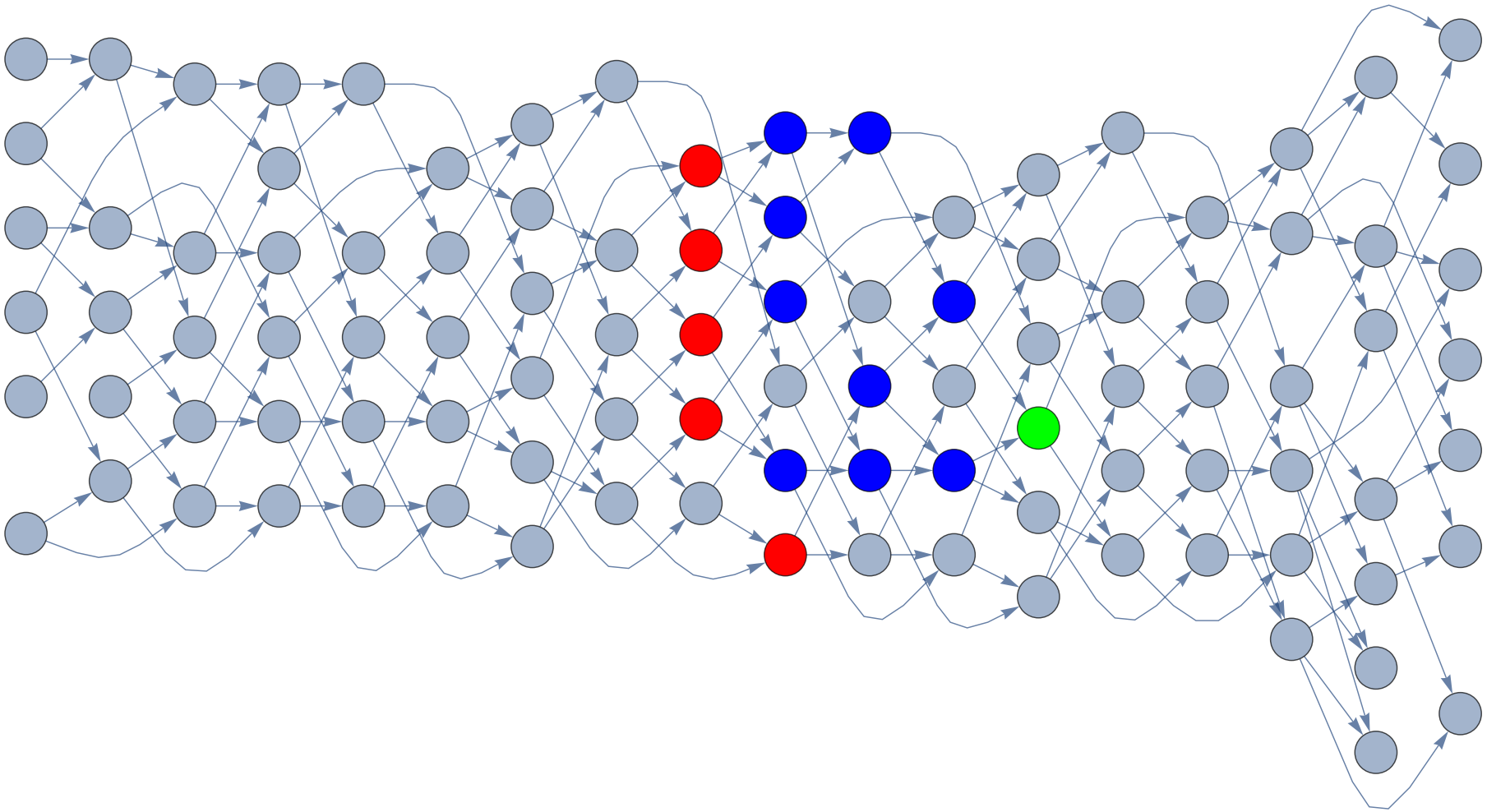
These correspond to AVX instructions **vpxor**, **vpand**, **vpor**, **vpandn**.

Rewriting

Many logic synthesis tools (e.g. Berkeley's ABC) work by **local rewriting**: replacing small subcircuits with more efficient equivalents.



Prior work

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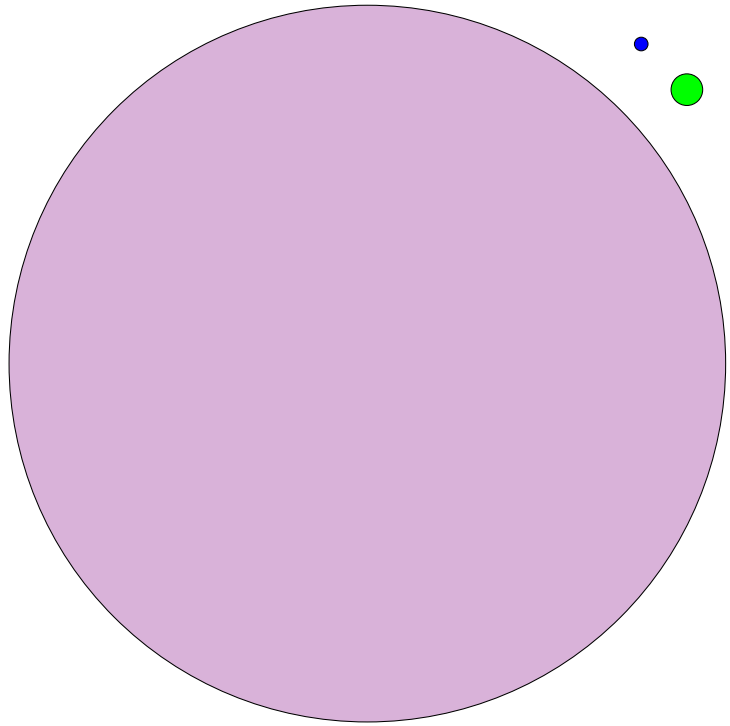

We shall take this to its ultimate logical conclusion: finding all optimal chains for **616125** of the 616126 equivalence classes of 5-input functions.

Equivalence classes

We can transform a k-input ℓ-output function into an equivalent function by:

- Permuting inputs ($k!$ possibilities);
- Negating inputs (2^k possibilities);
- Permuting outputs ($\ell!$ possibilities);
- Negating outputs (2^ℓ possibilities);

which generate the group $(S_2 \wr S_k) \times (S_2 \wr S_\ell)$ of order $2^{k+\ell}(k!)(\ell!)$.

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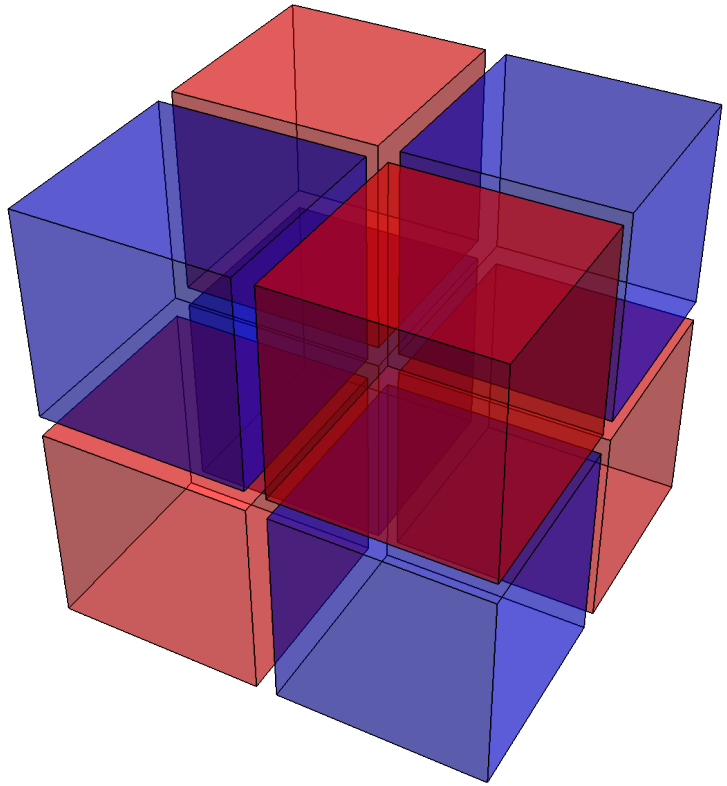
$$0001 < 0010 < 0100 < 0111 < 1000 < 1011 < 1101 < 1110$$

We describe the lexicographically first truth table in an equivalence class as **canonical**.

Counting equivalence classes



We can count the equivalence classes using Burnside's lemma; the dominant term will be:

$$|C| \approxeq \frac{2^{2^k \ell}}{2^{k+\ell}(k!)(\ell!)}$$

Number of classes of functions of each cost

n	5-input 1-output	4-input 2-output
0	2	4
1	2	8
2	5	38
3	20	193
4	93	916
5	389	4869
6	1988	27219
7	11382	135402
8	60713	475926
9	221541	713796
10	293455	117828
11	26535	19
12	1	0
Total	**616126**	**1476218**

Size of search space

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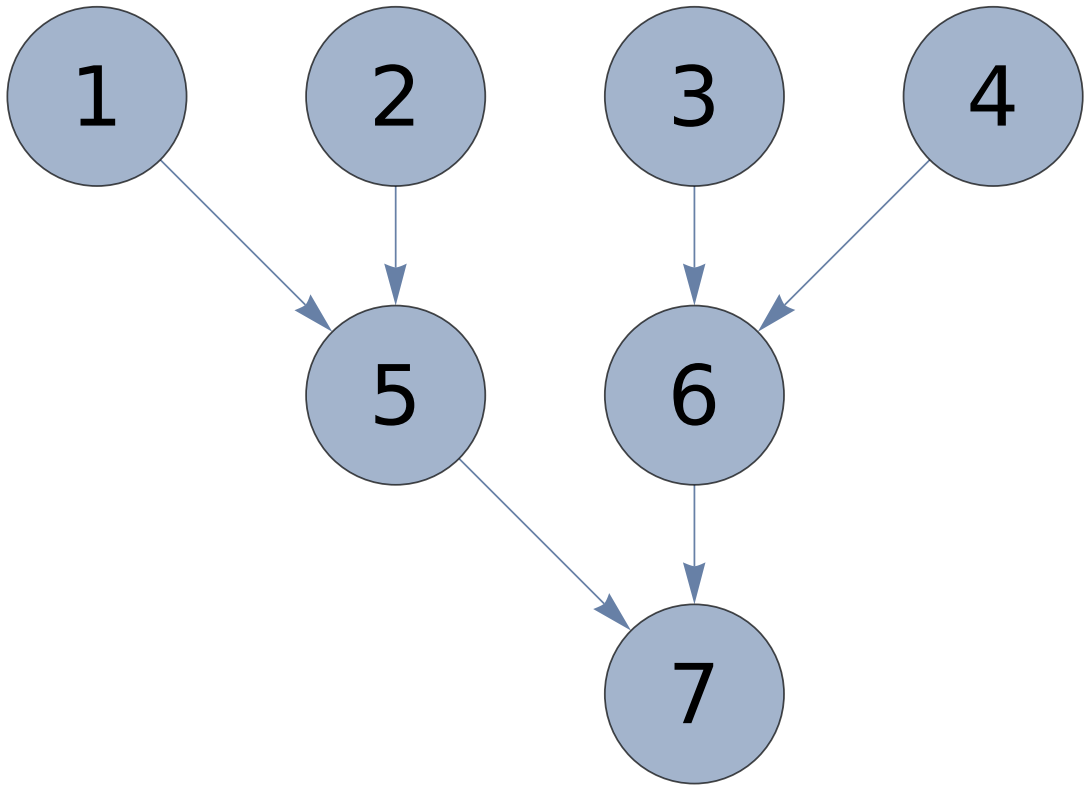
This is about **60x more** than the number of floating-point operations used to train GPT-3 (3.14×10^{23}).

Room for improvement I: canonical ordering

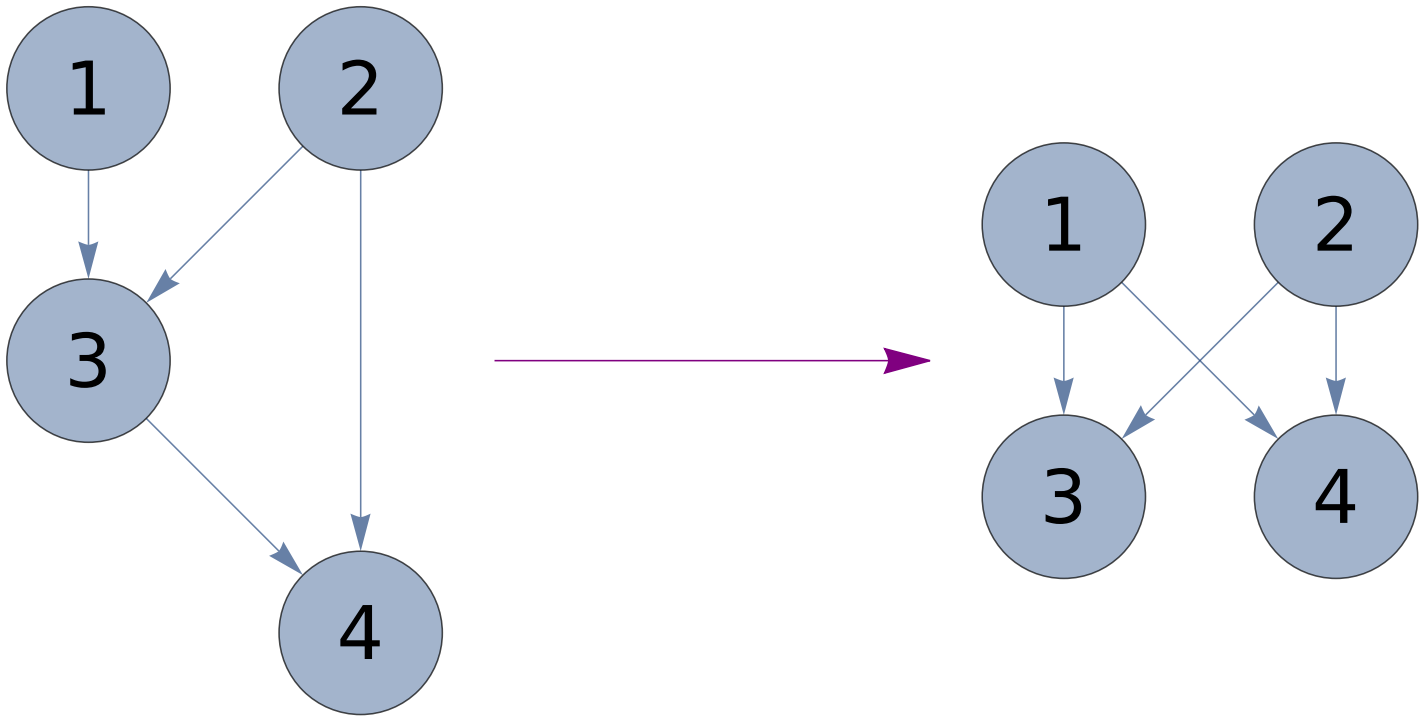
In many cases, a single DAG can correspond to multiple Boolean chains:



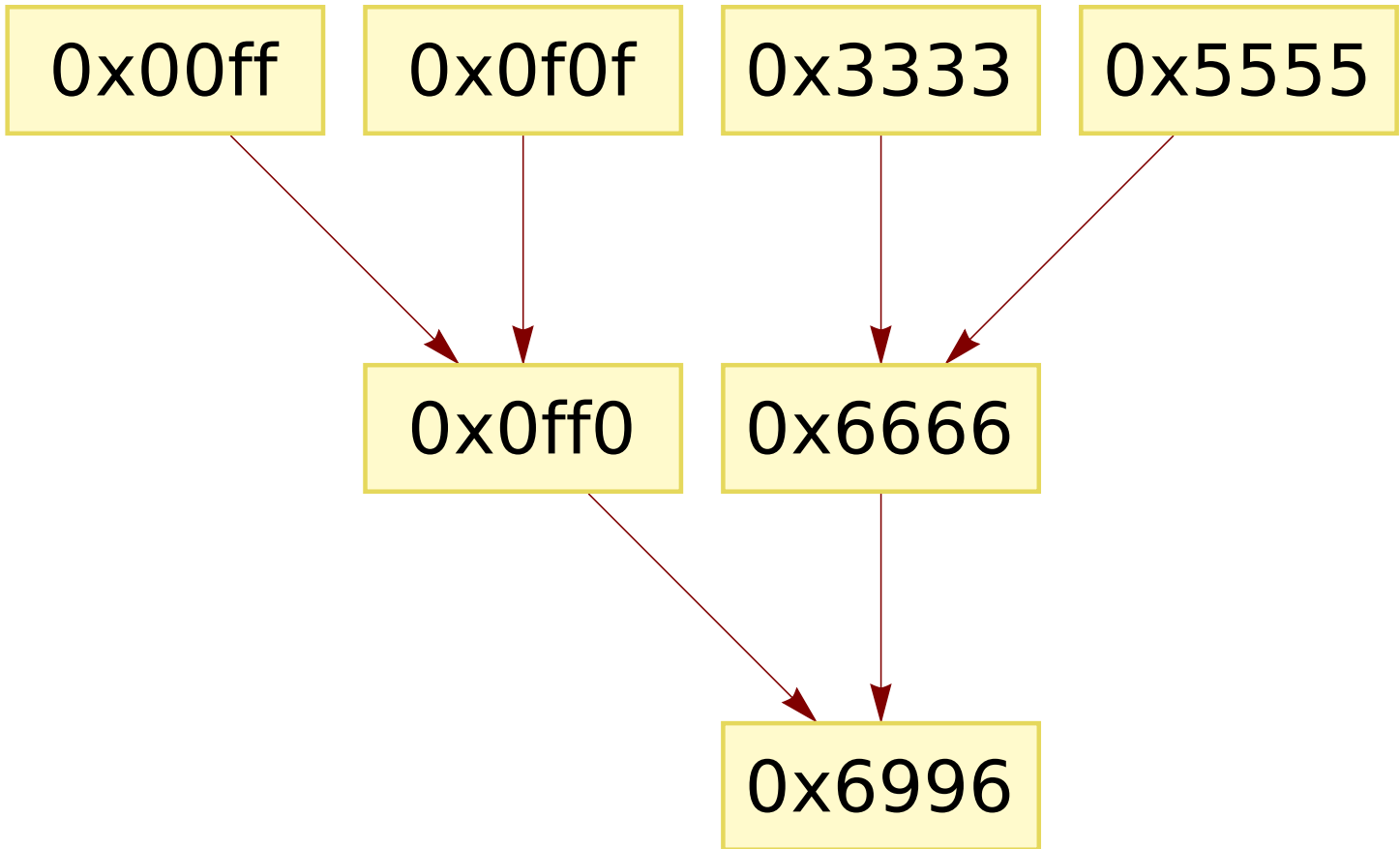
For example, the order of computing variables 5 and 6 could be swapped.

Room for improvement II: triangle-free

Also, triangles can be removed from our DAG without loss of generality:



Key insight



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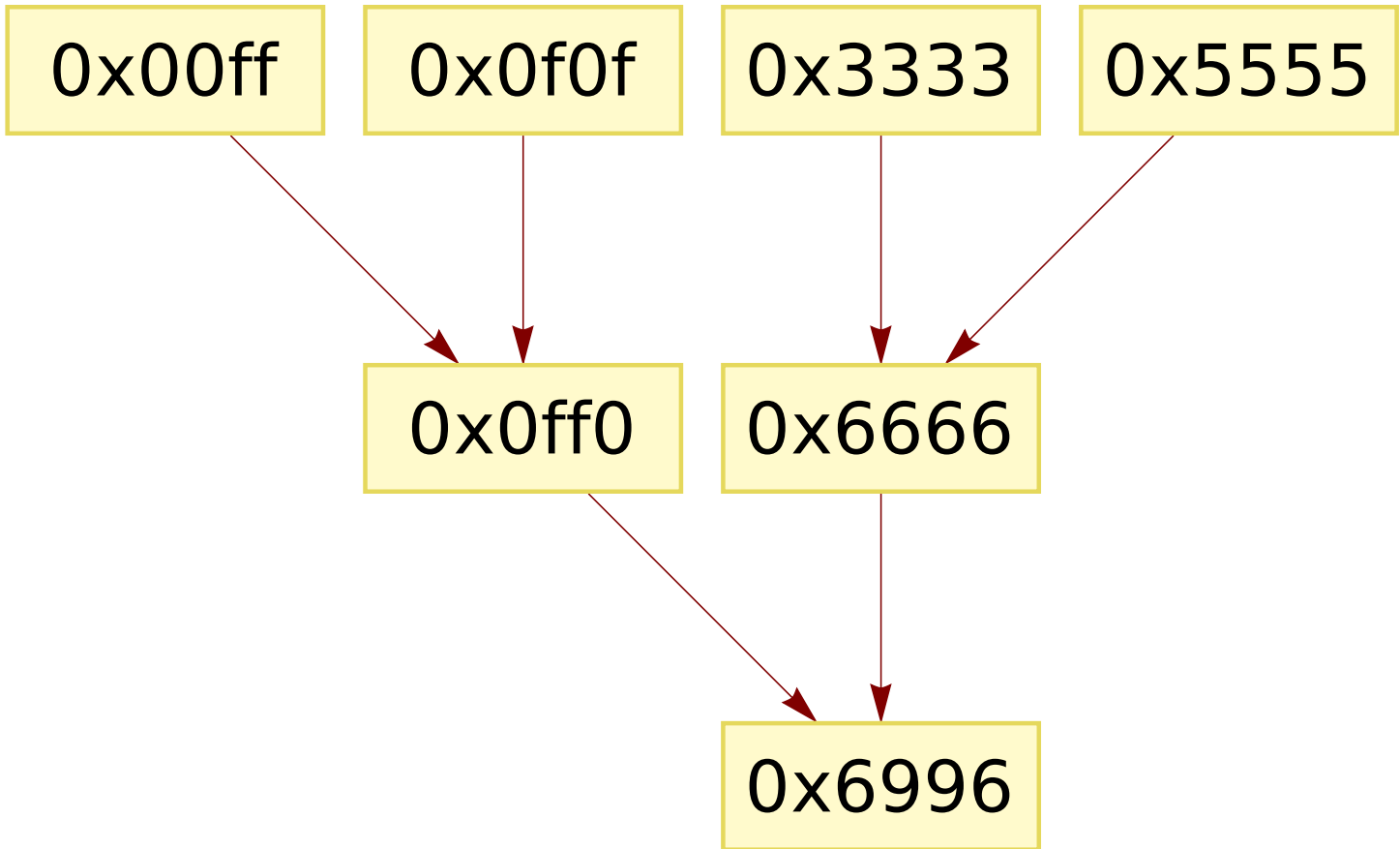

Simply represent this as the set {`0x0ff0,0x6666,0x6996`}.

Reconstruction of chains from tt-sets

By performing a **backtracking search**, we can reconstruct all possible chains corresponding to a tt-set.

- Beginning with the set of truth tables corresponding to inputs, try applying all possible gates which result in a truth table belonging to the tt-set.

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- We enforce the 'canonical ordering' and 'triangle free' properties at all times.
- The total runtime is $O(Sn^3)$ where S is the number of chains outputted by the algorithm.

Ends of tt-sets

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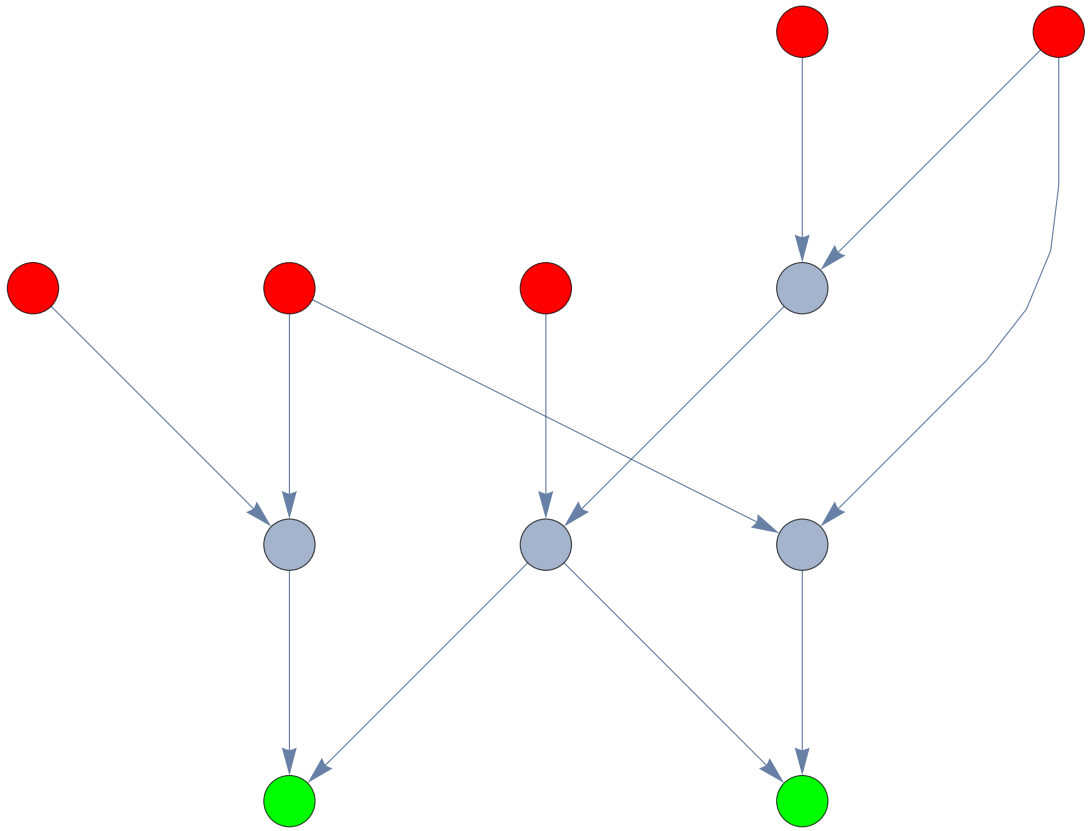
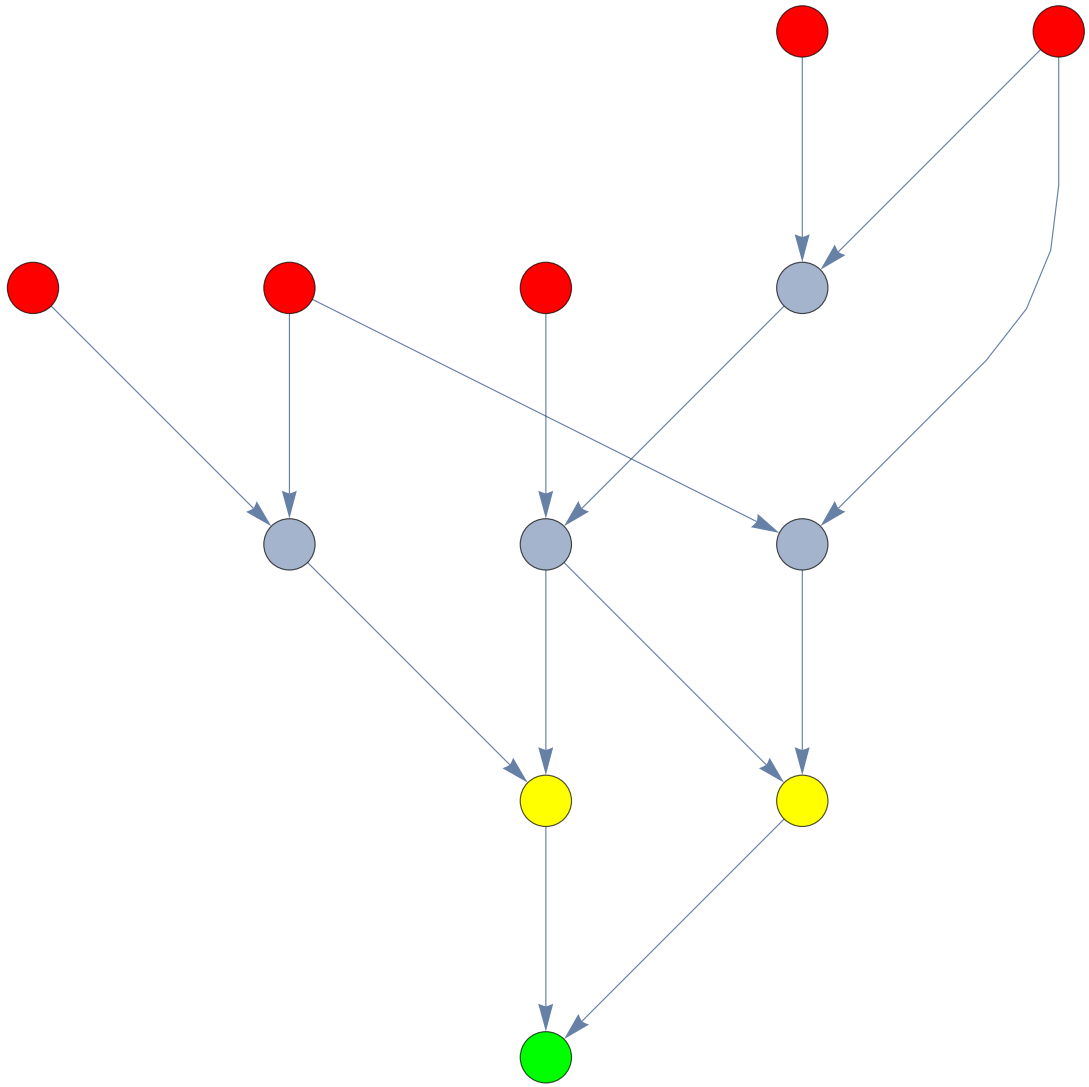


If $T \setminus \{e\}$ is still attainable, then we say that e is an *end* of T.

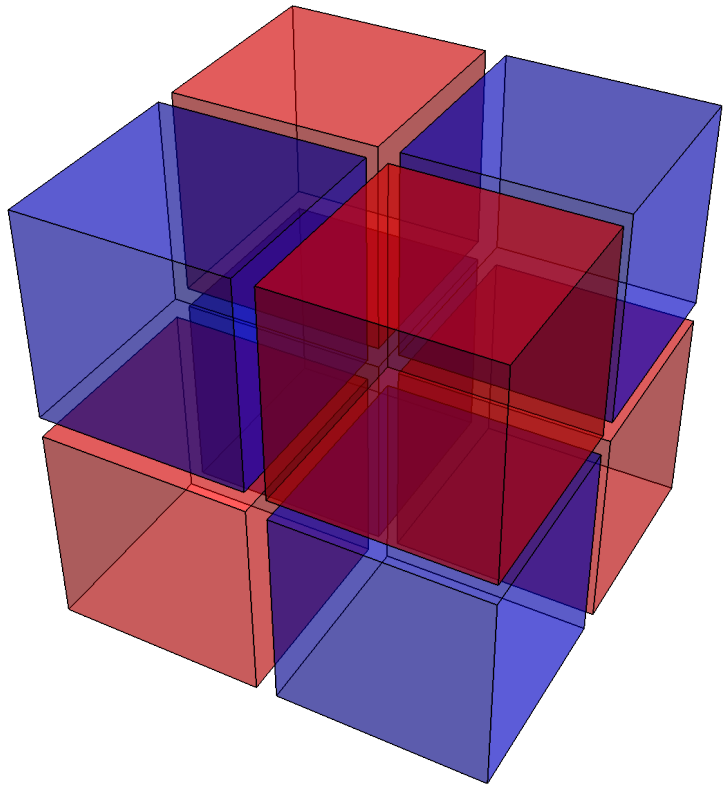
Appending a gate

Appending a gate cannot cause the number of ends to decrease by more than 1.



Symmetries

Recall that we have a large symmetry group at play.



The search becomes much faster if we only store **canonical** tt-sets.

Breadth-first search

Define $N(T)$ to be the set of possible 'next functions':

$$N(T) = \{x \circ y : x, y \in T \cup I \text{ and } \circ \in \mathcal{O}\} \setminus (T \cup I \cup \{0\})$$

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We construct all canonical attainable tt-sets by induction on size:

- Define $X_0 = \{\{\}\}$ to be the set containing the empty tt-set.
- Define
  $X_{n+1} = \{\text{canonicalise}(T \cup \{e\}) : T \in X_n \text{ and } e \in N(T)\}$

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For a given search depth, we can also discard tt-sets with too many ends.

How does previous work differ?

Previous approaches directly searched the (much larger) space of DAGs rather than canonical attainable tt-sets.

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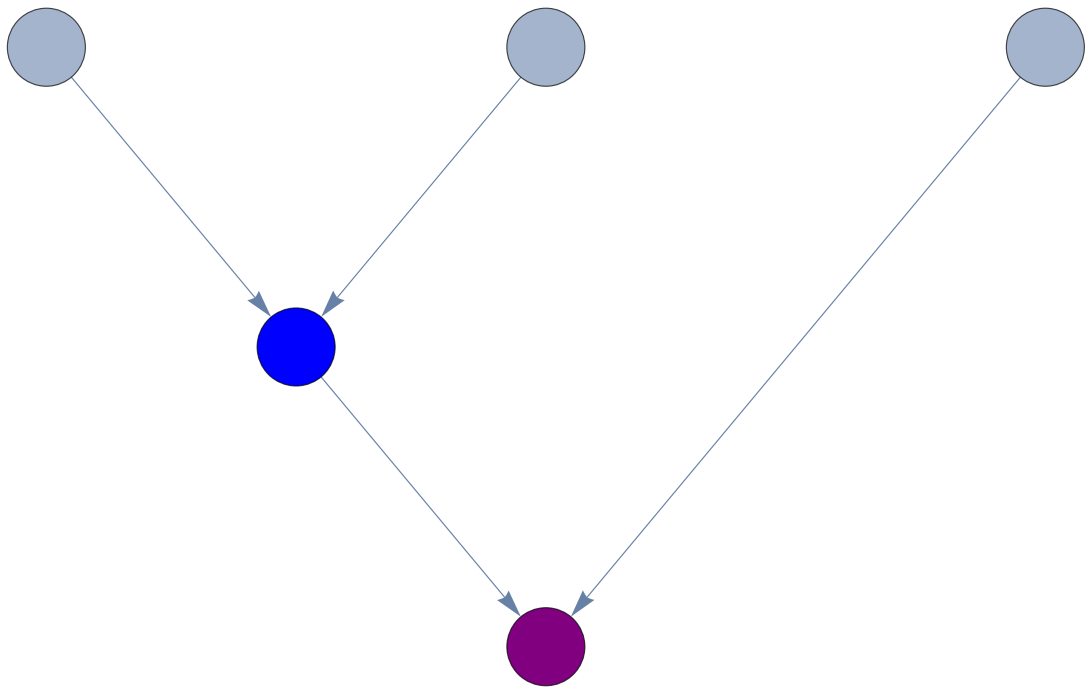
Haaswijk, Soeken, Mishchenko, and De Micheli (2018) used SAT solvers to search for DAGs of a particular topology implementing a given function.

Reducing constant factors

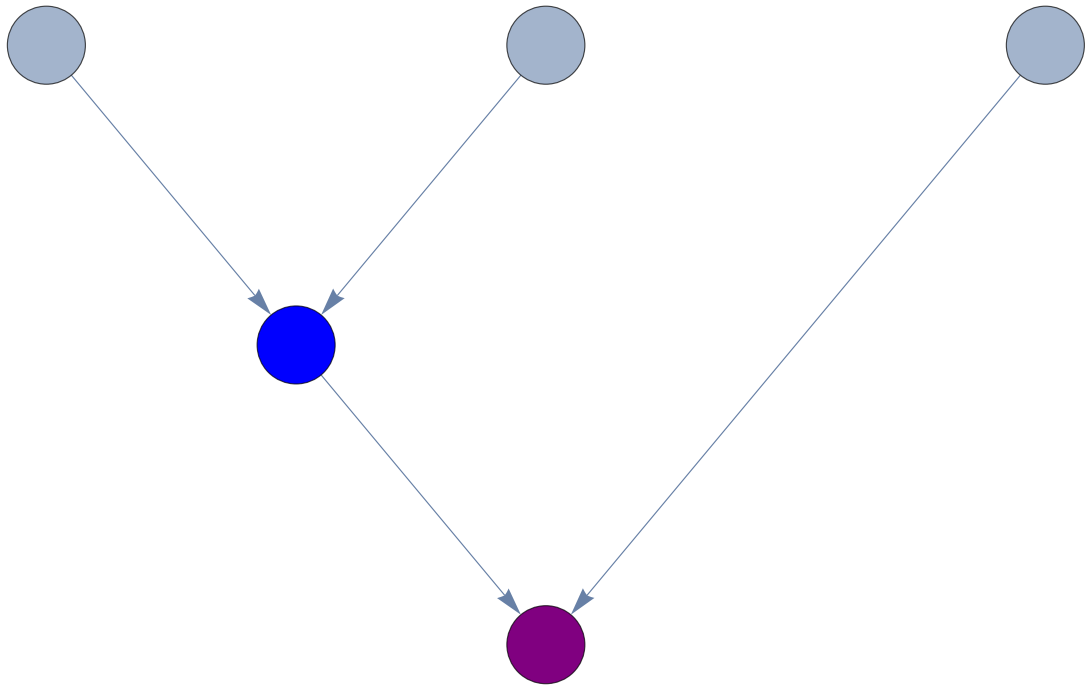
Canonicalising a tt-set is expensive. To do an 11-gate search, we do the following:

- Use the algorithm to compute all tt-sets in X_9 with ≤ 3 ends.
- Brute-force all possibilities for the last two gates.



Checking optimality

We want to check optimality, i.e. determine whether the purple vertex cannot be attained with fewer gates.



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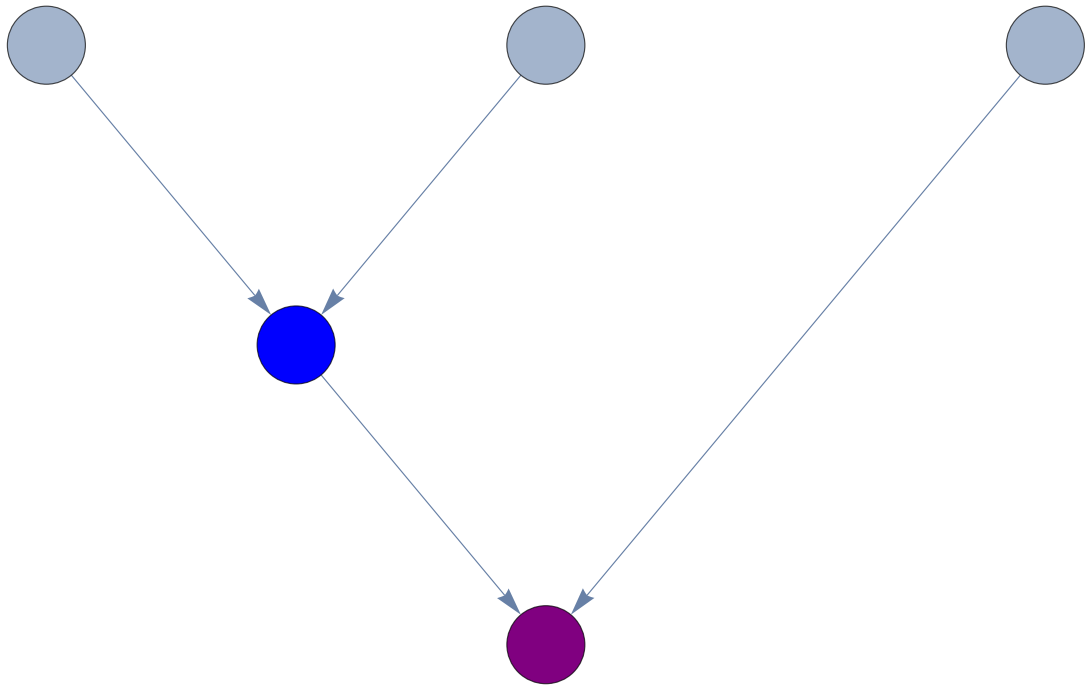

We use a 256 MB array (one bit for each of the 2^{31} normal functions) to indicate whether they've previously been constructed with lower cost.

Practical problems

Even a depth-9 search does not fit in memory (11 terabytes uncompressed).

n	tt-sets	uncompressed size	compressed size
6	3 million	73 MB	5 MB
7	130 million	3.6 GB	159 MB
8	6.2 billion	197 GB	6.4 GB
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We need to somehow **partition the search space** into manageable chunks, but this is not particularly easy.

Invariants and signatures

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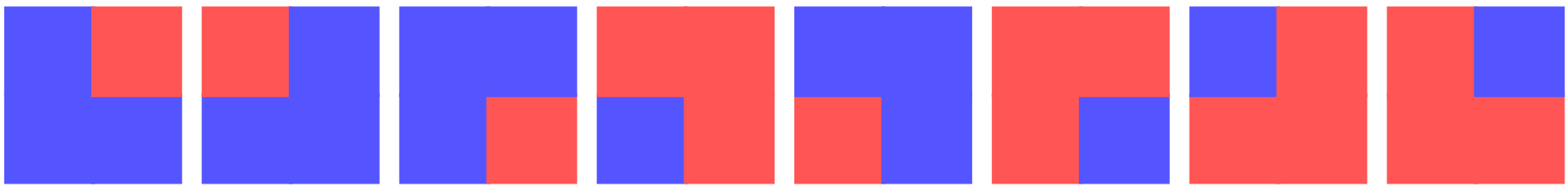
For example, the **bias**, or absolute difference between the number of '1's and '0's in the truth table, is an invariant.



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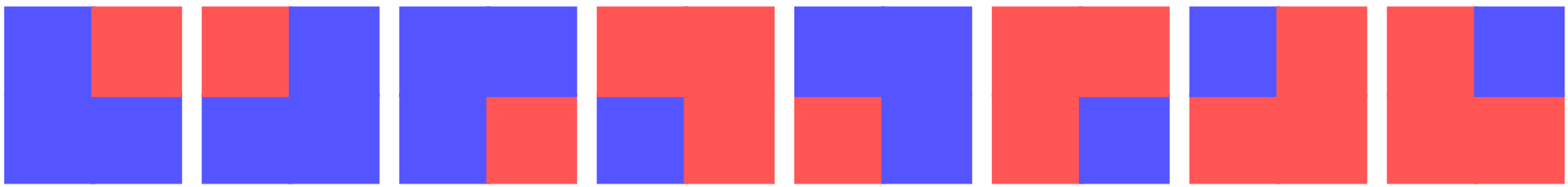

Given an invariant f, the **signature** of a tt-set T is the **multiset** $\{f(x) : x \in T\}$.

Distributing the workload

Proceed as before, but partition the creation of each X_n into separate tasks, one for each size-n signature.

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We avoid saving tt-sets to disk in the last level of the search tree ($n = 9$), because there are no downstream tasks to consume them.

Results

We applied this depth-11 exhaustive search procedure to two search problems:

- 5-input 1-output functions (616126 equivalence classes);
- 4-input 2-output functions (1476218 equivalence classes).

Each of the two searches took a few days on an AWS **r5a.24xlarge** instance (96 virtual cores + 768 GB memory), costing $< \$1000$.

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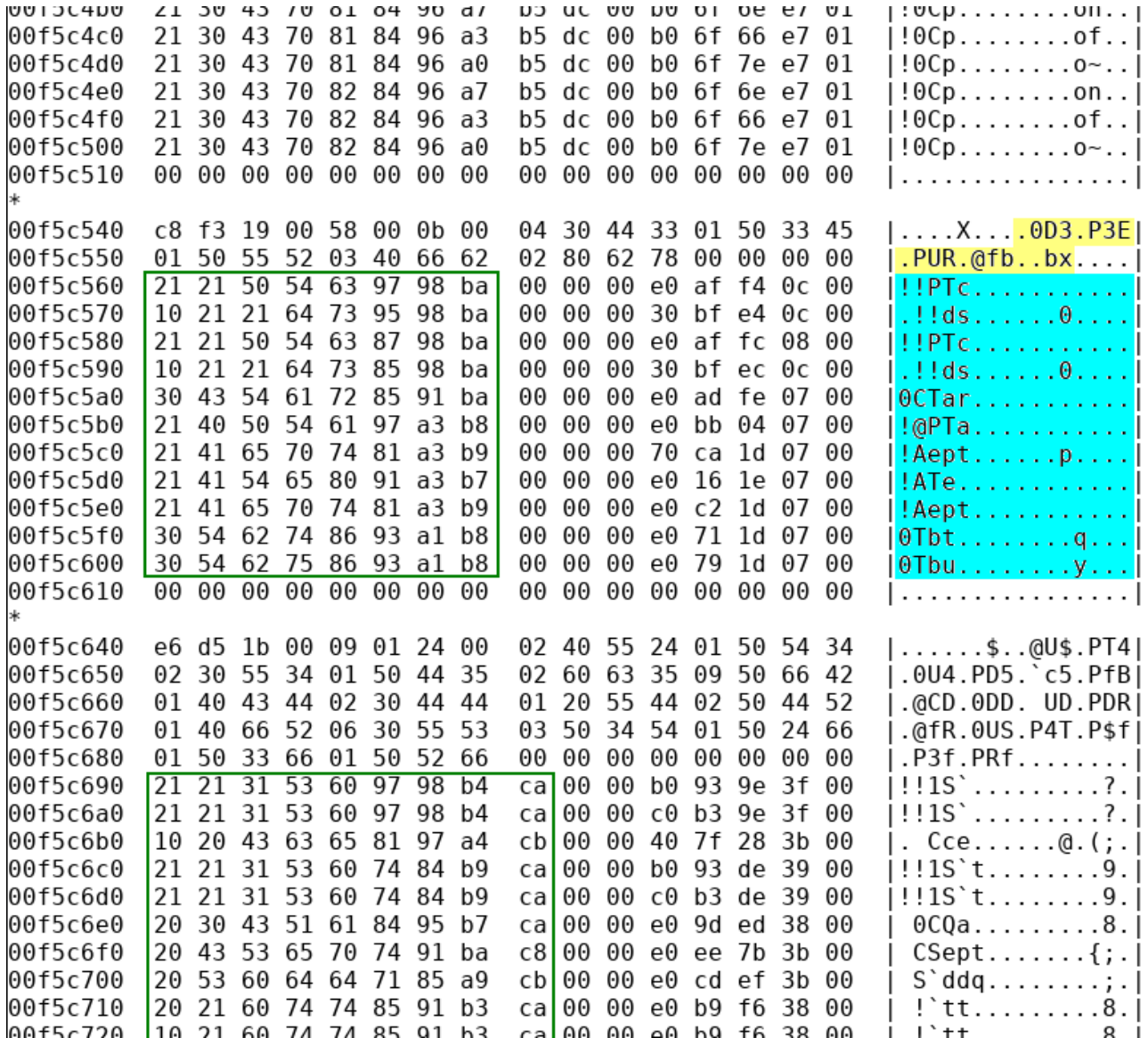
But how do we use these results?

Building a database

For each equivalence class of functions, we took all Boolean chains with **minimum cost** and **delay on the Pareto frontier**.



The k-perfect hashtable

We hash a canonical 5-input 1-output truth table using the following pair of functions:

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auto h1 = (x ^ (x >> 7) ^ (x >> 4)) & 0x1ffff;
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The function h_1 is 15-perfect, mapping at most 15 of the 616124 canonical nontrivial truth tables to the same bucket.

The function h_2 is 1-perfect (injective) within each bucket.

Looking up a canonical truth table

By squeezing each bucket into 64 bytes, it is possible to lookup the cost and location of any canonical function by retrieving a single cache line:

```
uint32_t get_cl_and_cost(const uint32_t* db, uint32_t x, int n_bits) {

    uint32_t clc = 0;
    uint32_t hash1 = hh::boolchains::hthash1(x, n_bits);
    uint32_t hash2 = hh::boolchains::hthash2(x);

    uint32_t cache_line[16];

    memcpy(cache_line, &(db[hash1 << 4]), 64);

    for (size_t i = 0; i < 15; i++) {
        uint32_t element = cache_line[i];
        if ((element & 0x1ff) == hash2) {
            uint32_t cl = ((element >> 9) & 0x7fff) + cache_line[15];
            uint32_t cost = (element >> 24);
            clc = (cl << 4) | cost;
            break;
        }
    }

    return clc;
}
```

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- **Lookup the cost and location**: this is a single cache line retrieval plus a handful of instructions, so takes around **100 ns**.
- **Iterate over optimal chains**: this step takes variable time, depending on the number of chains, but they're contiguous in memory so this is again reasonably fast.